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Global solutions of initial value problems for nonlinear second-order integro-differential equations of mixed type in Banach spaces ☆

Hua Su [a,b], Lishan Liu [a,c,*], Xiaoyan Zhang [b], Yonghong Wu [c]

[a] *Department of Mathematics, Qufu Normal University, Qufu 273165, PR China*
[b] *School of Mathematics and System Sciences, Shandong University, Jinan 250100, PR China*
[c] *Department of Mathematics and Statistics, Curtin University of Technology, Box 6845, Perth, Australia*

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Abstract

In this paper, the global solutions of initial value problems for nonlinear second-order integro-differential equations of mixed type in Banach spaces are investigated. The existence and uniqueness of solutions and their iterative approximation are obtained by using Mönch fixed point theorem and new comparison results. The results presented here essentially improve, generalize and unify many well-known results.

Keywords: Second-order integro-differential equations of mixed type; Measure of non-compactness; Global solutions; Fixed point; Banach space

1. Introduction

In this paper, we consider the existence of solutions and iterative approximation of the unique solution for the following initial value problems (IVP) of nonlinear second-order integro-differential equations of mixed type in ordered Banach spaces E

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* Corresponding author. Fax: +86 537 4455076.
E-mail addresses: jnsuhua@163.com (H. Su), lls@mail.qfnu.edu.cn (L. Liu), yhwu@maths.curtin.edu.au (Y. Wu).

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$$\begin{cases} u'' = f(t, u, u', Tu, Su), & t \in J, \\ u(0) = x_0, & u'(0) = x_1, \end{cases} \tag{1.1}$$

where $J = [0, a]$ $(0 < a \leqslant 1)$, $x_0, x_1 \in E$, $f \in C[J \times E \times E \times E \times E, E]$, and

$$(Tu)(t) = \int_0^t k(t,s)u(s)\,ds, \qquad (Su)(t) = \int_0^a h(t,s)u(s)\,ds. \tag{1.2}$$

In (1.2), $k(t,s) \in C[D, \mathbf{R}^+]$, $h(t,s) \in C[D_0, \mathbf{R}^+]$, where $D = \{(t,s) \in \mathbf{R} \times \mathbf{R} \mid 0 \leqslant s \leqslant t \leqslant a\}$, $D_0 = \{(t,s) \in \mathbf{R} \times \mathbf{R} \mid (t,s) \in J \times J\}$, $\mathbf{R}^+ = [0, +\infty)$.

The solutions for initial value problems (IVP) of nonlinear first-order integro-differential equations of mixed type in ordered Banach spaces have been obtained by means of the mixed monotone iterative technique in [1–3]. But there is a little discussion for the solutions of (IVP) (1.1). Recently, in the special case where f does not contain differential argument u', Guo [4] discussed the minimal and maximal solutions of (IVP) (1.1) and the monotone iterative sequence with the stronger conditions. And in another special case where f does not contain the operator Su, Song [6] obtained the minimal or maximal solutions of (IVP) (1.1), the unique solutions by means of the lower or upper solutions and the mixed monotone iterative technique with the stronger conditions.

In this paper, by using some new comparison results and Mönch fixed point theorem, we researched the more general mixed type case (IVP) (1.1) where f contains u', Su and obtained its global solutions and unique solutions as well as its iterative approximation under the conditions which are more extensive than those in [3–6]. The results presented in this paper essentially improve, generalize and unify many known results (see [3–6]). Our method is different in essence from those of [3–6].

2. Preliminaries and lemmas

In this paper, we always suppose that $(E, \|\cdot\|)$ is a real Banach space. A nonempty closed convex subset P in E is said to be a cone if $\lambda P \in P$ for $\lambda \geqslant 0$ and $P \cap \{-P\} = \{\theta\}$, where θ denotes the zero element of E. The cone P defines a partial ordering in E by $x \leqslant y$ iff $y - x \in P$. Recall the cone P is said to be normal if there exists a positive constant λ such that $0 \leqslant x \leqslant y$ implies $\|x\| \leqslant \lambda \|y\|$. The cone P is normal if every order interval $[x, y] = \{z \in E \mid x \leqslant z \leqslant y\}$ is bounded in norm. Let

$$C[J,E] = \{u : J \to E \mid u(t) \text{ is continuous}\},$$
$$C^1[J,E] = \{u : J \to E \mid u(t) \text{ is continuously differentiable}\},$$
$$C^2[J,E] = \{u : J \to E \mid u(t) \text{ is second-order continuously differentiable}\}.$$

For $u_0, v_0 \in C^2[J, E]$, let $G = \{u \in C^1[J, E] \mid u_0 \leqslant u \leqslant v_0,\ u_0' \leqslant u' \leqslant v_0'\}$. For $u = u(t) \in C[J, E]$, let $\|u\|_C = \max_{t \in J} \|u(t)\|$, then $C[J, E]$ is a Banach space with the norm $\|\cdot\|_C$. For $u = u(t) \in C^1[J, E]$, then $u' \in C[J, E]$, let $\|u\|_{C^1} = \max\{\|u\|_C, \|u'\|_C\}$, it is easy to see that $C^1[J, E]$ is a Banach space with the norm $\|\cdot\|_{C^1}$. Let $P_C = \{u(t) \in C[J, E] \mid u(t) \geqslant \theta,\ t \in J\}$, obviously the normality of P implies the normality P_C and the normal constants of P_C and P are the same. For further details on the cone theory, one can refer to see [6,7].

Given a cone P, the dual of P is defined as $P^* = \{\varphi \in E^* \mid \varphi(x) \geqslant 0,\ x \in P\}$, where E^* is the dual of E. Let $k_0 = \max\{k(t,s) \mid (t,s) \in D\}$, $h_0 = \max\{h(t,s) \mid (t,s) \in D_0\}$. For any $B \in C[J, E]$, and $t \in J$, let

$$B(t)=\{u(t) \mid u\in B\},\qquad (TB)(t)=\{(Tu)(t) \mid u\in B\},$$
$$(SB)(t)=\{(Su)(t) \mid u\in B\}.$$

The following comparison results play an important role in this paper.

Lemma 2.1. *Assume that E is a Banach space, P is a cone in E and $u=u(t)\in C^2[J,E]$ satisfies*

$$\begin{cases} u''(t)\geqslant -M(t)u(t)-N(t)u'(t)-L(t)(Tu)(t), & t\in J,\\ u(0)=\theta, \qquad u'(0)\geqslant\theta, \end{cases} \tag{2.1}$$

where $M(t)$, $N(t)$, $L(t)$ are bounded integrable non-negative functions on J, and provided one of the following two conditions hold:

(i) $2a(aM+N+a^2Lk_0)<1$,
(ii) $N>0, \alpha(e^{Na}-1)(M+aLk_0)<N$,

here $M=\sup\{M(t) \mid t\in J\}$, $N=\sup\{N(t) \mid t\in J\}$, $L=\sup\{L(t) \mid t\in J\}$. Then $u(t)\geqslant\theta$, $u'(t)\geqslant\theta$, $\forall\, t\in J$.

Proof. For any $g\in P^*$, let $p(t)=g(u(t))$, $\forall t\in J$, then $p\in C^2[J,\mathbf{R}]$ and

$$p''(t)=g\big(u''(t)\big),\quad p'(t)=g\big(u'(t)\big),$$
$$(Tp)(t)=g\big((Tu)(t)\big),\quad (Sp)(t)=g\big((Su)(t)\big),\quad \forall t\in J.$$

By (2.1), we have

$$\begin{cases} p''(t)\geqslant -M(t)p(t)-N(t)p'(t)-L(t)(Tp)(t), & \forall\, t\in J,\\ p(0)=0, \qquad p'(0)\geqslant 0. \end{cases} \tag{2.2}$$

Let $p_1(t)=p'(t)$, then $p_1(t)\in C^1[J,\mathbf{R}]$, and $p(t)=p(0)+\int_0^t p_1(s)\,ds=\int_0^t p_1(s)\,ds$. Therefore, by (2.2), we have

$$\begin{cases} p_1'(t)\geqslant -\int_0^t\left[M(s)+L(s)\int_s^t k(t,r)dr\right]p_1(s)\,ds-N(t)p_1(t), & \forall\, t\in J,\\ p_1(0)\geqslant 0. \end{cases} \tag{2.3}$$

We shall show that $p_1(t)\geqslant 0$, $\forall t\in J$. In fact, if we suppose $p_1(t)\geqslant 0$ is not true, then there exists a $t_0\in(0,a]$ such that $p_1(t_0)<0$. Let $\max\{p_1(t)\colon 0\leqslant t\leqslant t_0\}=\lambda$, then $\lambda\geqslant 0$.

If $\lambda=0$, then $p_1(t)\leqslant 0$, $\forall t\in[0,t_0]$. So by (2.3), we have $p_1'(t)\geqslant 0$, $\forall t\in[0,t_0]$. Consequently, $p_1(t)$ is increasing in $[0,t_0]$, so $p_1(t_0)\geqslant p_1(0)\geqslant 0$, which contradicts $p_1(t_0)<0$.

If $\lambda>0$, there exists a $t_1\in[0,t_0]$ such that $p_1(t_1)=\lambda$. From (2.3), we have

$$p_1'(t)\geqslant -a[M+Lk_0a]\lambda-N\lambda=-\big[aM+N+Lk_0a^2\big]\lambda,\quad \forall t\in[0,t_0].$$

By mean value theorem, there exists a $t_2\in(t_1,t_0)$ such that

$$\begin{aligned} p_1(t_0)&=p_1(t_1)+(t_0-t_1)p_1'(t_2)\geqslant p_1(t_1)-a\big[aM+N+Lk_0a^2\big]\lambda\\ &=\lambda-a\big[aM+N+Lk_0a^2\big]\lambda. \end{aligned}$$

Then, by $p_1(t_0)<0$, we have $a[aM+N+Lk_0a^2]\geqslant 1$, which contradicts condition (i).

In case of condition (ii) holding, let

$$w(t) = p_1(t)e^{\int_0^t N(s)\,ds},$$

apply it to (2.3), we have

$$\begin{cases} w'(t) \geqslant -\displaystyle\int_0^t \left[M(s) + L(s)\int_s^t k(t,r)\,dr\right] e^{\int_s^t N(\xi)d\xi} w(s)\,ds, & \forall t \in J, \\ w(0) \geqslant 0. \end{cases}$$

By the similar proof process to above, we can obtain $w(t) \geqslant 0$, $\forall t \in J$. And so $p_1(t) \geqslant 0$, $\forall t \in J$. Therefore, we have $p_1(t) \geqslant 0$, $\forall t \in J$, i.e., $p'(t) \geqslant 0$, $\forall t \in J$. And so $p(t) \geqslant p(0) = 0$, $\forall t \in J$. By the randomness of $g \in p^*$, we know $u(t) \geqslant \theta$, $u'(t) \geqslant \theta$, $\forall t \in J$. This completes the proof. □

Lemma 2.2. [7–9] *Assume that $B \in C^1[J, E]$, and B' is equicontinuous. Then*

$$\alpha_1(B) = \max\Big\{\max_{t\in J} \alpha\big(B(t)\big),\ \max_{t\in J} \alpha\big(B'(t)\big)\Big\},$$

where $\alpha_1(\cdot)$ denotes the Kuratowski measure of non-compactness in $C^1[J, E]$.

Lemma 2.3. [1] *Let $B_1, B_2 \subset C[J, E]$ be two countable subset satisfying $\overline{B}_1 = \overline{co}(\{u_0\} \cup B_2)$. for some $u_0 \in C[J, E]$. Then $\overline{B_1(t)} = \overline{co}(\{u_0(t)\} \cup B_2(t))$ for any $t \in J$.*

Lemma 2.4. [7–9] *Let $B \subset C[J, E]$ be bounded and equicontinuous. Define $m(t) = \alpha(B(t))$, $t \in J$. Then $m(t)$ is continuous on J and*

$$\alpha\bigg(\int_J B(s)\,ds\bigg) \leqslant \int_J \alpha\big(B(s)\big)\,ds.$$

Lemma 2.5. [1,2] *Assume that $m \in C[J, \mathbf{R}^+]$ satisfies*

$$m(t) \leqslant M_1 \int_0^t m(s)\,ds + M_2 t \int_0^t m(s)\,ds + M_3 t \int_0^a m(s)\,ds, \quad t \in J,$$

where $M_1 > 0$, $M_2 \geqslant 0$, $M_3 \geqslant 0$ are constants. Then $m(t) \equiv 0$ for any $t \in J$, provided one of the following two conditions holds:

(i) $aM_3(e^{a(M_1+aM_2)} - 1) < M_1 + aM_2$,
(ii) $a(2M_1 + aM_2 + aM_3) < 2$.

Lemma 2.6. [7,10] *Let E be a Banach space, $K \subset E$ closed and convex and $F : K \to K$ continuous with the further property that for $x_0 \in K$ we have*

$$C \subset K \quad \text{countable}, \qquad \overline{C} = \overline{co}\big(\{x_0\} \cup F(C)\big) \quad \Longrightarrow \quad C \text{ is relatively compact}.$$

Then F has a fixed point in K.

Lemma 2.7. [7,8] *Let $B \subset C[J,E]$ be a countable set of strongly measurable functions $u: J \to E$ such that there exists an $m \in L[J, \mathbf{R}^+]$ such that $\|u(t)\| \leqslant m(t)$ for any $t \in J$ and $u \in B$. Then $\alpha(B(t)) \in L[J, \mathbf{R}^+]$ and*

$$\alpha\left(\left\{\int_J u(t)\,dt \,\middle|\, u \in B\right\}\right) \leqslant 2\int_J \alpha\big(B(t)\big)t.$$

Let us list the following assumption for convenience.

(H_1) There exist $u_0, v_0 \in C^2[J,E]$ such that $u_0 \leqslant v_0$, $u_0' \leqslant v_0'$, and bounded integrable non-negative functions $M(t)$, $N(t)$, $L(t)$ on J which satisfy (i) or (ii) in Lemma 2.1, for any $\sigma \in G$, $t \in J$,

$$\begin{cases} u_0''(t) \leqslant f(t,\sigma,\sigma',T\sigma,S\sigma) - M(t)(u_0-\sigma) - N(t)(u_0'-\sigma') - L(t)T(u_0-\sigma), \\ \quad t \in J, \\ u_0(0) = x_0, \qquad u_0'(0) \leqslant x_1; \end{cases}$$

$$\begin{cases} v_0''(t) \geqslant f(t,\sigma,\sigma',T\sigma,S\sigma) - M(t)(v_0-\sigma) - N(t)(v_0'-\sigma') - L(t)T(v_0-\sigma), \\ \quad t \in J, \\ v_0(0) = x_0, \qquad v_0'(0) \geqslant x_1. \end{cases}$$

(H_2) For any countable bounded equicontinuous set $B = \{u_n\} \subset G$ and $t \in J$,

$$\begin{aligned} &\alpha\big(f\big(t, B(t), B'(t), (TB)(t), (SB)(t)\big)\big) \\ &\quad \leqslant r_1(t)\alpha\big(B(t)\big) + r_2(t)\alpha\big(B'(t)\big) + r_3(t)\alpha\big((TB)(t)\big) + r_4(t)\alpha\big((SB)(t)\big), \end{aligned}$$

where $r_i(t)$ $(i = 1,2,3,4)$ are bounded integrable non-negative functions satisfying one of the following two conditions:

(i) $2r_4h_0(e^{2a(a+1)[(r_1+M+r_2+N)+a(L+r_3)k_0]} - 1) < [(r_1+M+r_2+N) + a(L+r_3)k_0]$,

(ii) $a(a+1)[2(r_1+M+r_2+N) + a(L+r_3)k_0 + ar_4h_0] < 1$, where $r_i = \sup\{r_i(t) \mid t \in J\}$ $(i = 1,2,3,4)$.

(H_3) For any $u_0 \leqslant x_i \leqslant v_0$, $u_0' \leqslant y_i \leqslant v_0'$, $Tu_0 \leqslant z_i \leqslant Tv_0$, $Su_0 \leqslant w_i \leqslant Sv_0$ and $t \in J$,

$$\begin{aligned} &\big\|f(t,x_1,y_1,Tz_1,Sw_1) - f(t,x_2,y_2,Tz_2,Sw_2)\big\| \\ &\quad \leqslant r_1(t)\|x_1-x_2\| + r_2(t)\|y_1-y_2\| + r_3(t)\|z_1-z_2\| + r_4(t)\|w_1-w_2\|, \end{aligned}$$

where $r_i(t)$ $(i = 1,2,3,4)$ are bounded integrable non-negative functions satisfying (i) or (ii) in (H_2).

3. The main results

At first we will prove the following theorem on global solutions of (IVP) (1.1).

Theorem 3.1. *Let E be a real Banach space and P be a normal cone in E. Assume that conditions (H_1), (H_2) hold, then (IVP) (1.1) has a solution $u^* \in G$.*

Proof. We will divide the rather long proof into five steps.

(I) For any $h \in G$, consider the following (IVP) of linear second-order integro-differential equation (LIVP) in E

$$\begin{cases} u''(t) = g(t) - M(t)u(t) - N(t)u'(t) - L(t)(Tu)(t), \quad t \in J, \\ u(0) = x_0, \qquad u'(0) = x_1, \end{cases} \tag{3.1}$$

where $g(t) = f(t, h(t), h'(t), (Th)(t), (Sh)(t)) + M(t)h(t) + N(t)h'(t) + L(t)(Th)(t)$, $t \in J$. We shall show that there exists a unique solution of (LIVP) (3.1). It is easy to check that $u \in C^2[J, E]$ is a solutions of (LIVP) (3.1) if and only if $u \in C^1[J, E]$ is a fixed point of the following operator:

$$(Au)(t) = x_0 + tx_1 + \int_0^t (t-s)\bigl[g(s) - M(s)u(s) - N(s)u'(s) - L(s)(Tu)(s)\bigr]\,ds. \tag{3.2}$$

For any $u, v \in C^1[J, E]$, $t \in J$, and from (3.2), we have

$$\begin{aligned}
&\bigl\|(Au)(t) - (Av)(t)\bigr\| \\
&\quad \leqslant \int_0^t a\left[M\bigl\|u(s) - v(s)\bigr\| + N\bigl\|u'(s) - v'(s)\bigr\| + Lk_0\int_0^s \bigl\|u(\tau) - v(\tau)\bigr\|\,d\tau\right]ds \\
&\quad \leqslant \int_0^t a\bigl[(M + aLk_0)\|u - v\|_C + N\|u' - v'\|_C\bigr]\,ds \\
&\quad \leqslant \|u - v\|_{C^1} a[M + N + aLk_0]t, \\
&\bigl\|(Au)'(t) - (Av)'(t)\bigr\| \\
&\quad \leqslant t\left[M\bigl\|u(s) - v(s)\bigr\| + N\bigl\|u'(s) - v'(s)\bigr\| + Lk_0\int_0^s \bigl\|u(\tau) - v(\tau)\bigr\|\,d\tau\right]ds \\
&\quad \leqslant t\bigl[(M + aLk_0)\|u - v\|_C + N\|u' - v'\|_C\bigr]\,ds \\
&\quad \leqslant \|u - v\|_{C^1}[M + N + aLk_0]t.
\end{aligned}$$

Then, for any $t \in J$, it is easy to show by induction that

$$\bigl\|(Au)^n(t) - (Av)^n(t)\bigr\| \leqslant \|u - v\|_{C^1} a^n[M + N + aLk_0]^n \frac{t^n}{n!}, \quad n = 1, 2, \ldots,$$

$$\bigl\|\bigl((Au)^n\bigr)'(t) - \bigl((Av)^n\bigr)'(t)\bigr\| \leqslant \|u - v\|_{C^1}[M + N + aLk_0]^n \frac{t^n}{n!}, \quad n = 1, 2, \ldots.$$

Hence

$$\bigl\|(Au)^n - (Av)^n\bigr\|_{C^1} \leqslant \|u - v\|_{C^1}\bigl(a^n + 1\bigr)[M + N + aLk_0]^n \frac{a^n}{n!}, \quad n = 1, 2, \ldots. \tag{3.3}$$

For $n_0 > 0$ large enough such that $(a^{n_0} + 1)[M + N + aLk_0]^{n_0} a^{n_0}/(n_0!) < 1$, we have A^{n_0} is a contraction in $C^1[J, E]$. So A has a unique fixed point $u_h \in C^1[J, E]$ and u_h is the unique solution of (LIVP) (3.1) in $C^2[J, E]$.

(II) Now for any $h \in G$, we can define an operator:

$$Bh = u_h, \tag{3.4}$$

where u_h is the unique solution of (LIVP) (3.1) corresponding to $h \in G$ which satisfies

$$\begin{cases} u_h'' = f\bigl(t, h(t), h'(t), (Th)(t), (Sh)(t)\bigr) - M(t)(u_h - h)(t) \\ \qquad - N(t)(u_h' - h')(t) - L(t)T(u_h - h)(t), \quad t \in J, \\ u_h(0) = x_0, \qquad u_h'(0) = x_1. \end{cases}$$

Then the operator B satisfies

$$Bh = x_0 + tx_1 + \int_0^t (t-s)\big[f\big(s, h(s), h'(s), (Th)(s), (Sh)(s)\big) - M(s)(Bh-h)(s) - N(s)\big((Bh)'-h'\big)(s) - L(s)T(Bh-h)(s)\big]\,ds, \tag{3.5}$$

evidently, for any $h \in G$, h is a solution of (IVP) (1.1) iff $h = Bh$, i.e., h is a fixed point of B.

(III) We will show that the operator $B : G \to G$. In fact, for any $h \in G$, let $u = Bh$. From the definition of B, we know that u satisfies (3.1). All we need to do is to prove $u_0 \leqslant u \leqslant v_0$, $u_0' \leqslant u' \leqslant v_0'$. For any $\varphi \in P^*$, let $p(t) = \varphi(u(t) - u_0(t))$, then $p(0) \geqslant 0$. For any $\sigma \in G$, by assumption (H_1), we know

$$\begin{aligned}
p(t)'' &= \varphi\big(u(t)'' - u_0(t)''\big) \\
&\geqslant \varphi\big[f\big(t, h(t), h'(t), (Th)(t), (Sh)(t)\big) - M(t)(u-h)(t) \\
&\quad - N(t)(u'-h')(t) - L(t)T(u-h)(t) - f(t, \sigma, \sigma', T\sigma, S\sigma) \\
&\quad + M(t)(u_0-\sigma)(t) + N(t)(u_0'-\sigma')(t) + L(t)T(u_0-\sigma)(t)\big].
\end{aligned}$$

Let $h = \sigma$ in above inequality, then

$$\begin{aligned}
p(t)'' &\geqslant -\varphi\big(M(t)(u-u_0)(t)\big) - \varphi\big(N(t)(u'-u_0')(t)\big) - L(t)T(u-u_0)(t) \\
&= -M(t)p(t) - N(t)p'(t) - L(t)(Tp)(t).
\end{aligned}$$

By Lemma 2.1, we know $p(t) \geqslant 0$, $p'(t) \geqslant 0$. From the randomness of $\varphi \in P^*$, we know $u \geqslant u_0$, $u' \geqslant u_0'$. By similar method we can obtain $u \leqslant v_0$, $u' \leqslant v_0'$. Then $B : G \to G$.

(IV) We will show that $B : G \to G$ is continuous. For any sequence $\{h_n\} \subset G$ and $h \in G$, with

$$\|h_n - h\|_{C^1} \to 0 \quad (n \to \infty). \tag{3.6}$$

By the compactness of interval $J = [0, a]$ and the continuity of f, it is easy to prove that

$$\big\|f\big(t, h_n(t), h_n'(t), (Th_n)(t), (Sh_n)(t)\big) - f\big(t, h(t), h'(t), (Th)(t), (Sh)(t)\big)\big\|_C \to 0 \quad (n \to \infty). \tag{3.7}$$

For any $t \in J$, (3.5) implies

$$\begin{aligned}
&\big\|Bh_n(t) - Bh(t)\big\| \\
&\quad\leqslant \int_0^t (t-s)\big[\big\|f\big(t, h_n(s), h_n'(s), (Th_n)(s), (Sh_n)(s)\big) \\
&\qquad - f\big(t, h(s), h'(s), (Th)(s), (Sh)(s)\big)\big\|\big]\,ds \\
&\qquad + a^2M\|h_n - h\|_C + a^2N\|h_n' - h'\|_C + a^3Lk_0\|h_n - h\|_C \\
&\qquad + a^2(M + aLk_0)\|Bh_n - Bh\|_C + a^2N\|Bh_n' - Bh'\|_C \\
&\quad\leqslant a^2\big\|f\big(t, h_n(t), h_n'(t), (Th_n)(t), (Sh_n)(t)\big) - f\big(t, h(t), h'(t), (Th)(t), (Sh)(t)\big)\big\|_C \\
&\qquad + a^2(M + N + Lk_0a)\|h_n - h\|_{C^1} + a^2(M + N + Lk_0a)\|Bh_n - Bh\|_{C^1},
\end{aligned}$$

then

$$\begin{aligned}\|Bh_n - Bh\|_C \leqslant{} & a^2 \big\| f(t, h_n, h_n', Th_n, Sh_n) - f(t, h, h', Th, Sh)\big\|_C \\ & + a^2(M + N + Lk_0a)\|h_n - h\|_{C^1} \\ & + a^2(M + N + Lk_0a)\|Bh_n - Bh\|_{C^1}.\end{aligned}$$

By similar method, we can obtain

$$\begin{aligned}\big\|(Bh_n)' - (Bh)'\big\|_C \leqslant{} & a \big\| f(t, h_n, h_n', Th_n, Sh_n) - f(t, h, h', Th, Sh)\big\|_C \\ & + a(M + N + Lk_0a)\|h_n - h\|_{C^1} \\ & + a(M + N + Lk_0a)\|Bh_n - Bh\|_{C^1}.\end{aligned}$$

Hence, we have

$$\begin{aligned}\|Bh_n - Bh\|_{C^1} \leqslant{} & a(a+1) \big\| f(t, h_n, h_n', Th_n, Sh_n) - f(t, h, h', Th, Sh)\big\|_C \\ & + a(a+1)(M + N + Lk_0a)\|h_n - h\|_{C^1} \\ & + a(a+1)(M + N + Lk_0a)\|Bh_n - Bh\|_{C^1} \\ \leqslant{} & 2\big\| f(t, h_n, h_n', Th_n, Sh_n) - f(t, h, h', Th, Sh)\big\|_C \\ & + 2(M + N + Lk_0a)\|h_n - h\|_{C^1} + 2(M + N + Lk_0)\|Bh_n - Bh\|_{C^1},\end{aligned}$$

i.e.,

$$\begin{aligned}& \|Bh_n - Bh\|_{C^1} \\ & \quad \leqslant \frac{1}{1 - 2(M + N + Lk_0)}\Big[2\big\| f(t, h_n, h_n', Th_n, Sh_n) - f(t, h, h', Th, Sh)\big\|_C \\ & \qquad + 2(M + N + Lk_0a)\|h_n - h\|_{C^1}\Big]. \qquad (3.8)\end{aligned}$$

From (3.6)–(3.8), we know that $B : G \to G$ is continuous.

(V) In the end we shall show that B has a fixed point in G. It is evident that $u \in G$ is a solutions of the (IVP) (1.1) iff u is a fixed point of B in G. For any $u \in G$, by (III), we have $u_0 \leqslant Bu \leqslant v_0$, $u_0' \leqslant (Bu)' \leqslant v_0'$. Let $K = \overline{co}(BG)$, then by step (IV), B is a continuous operator from K into K. It is easy to see that the normality of P implies the normality of P_C, and so, for any $h \in G$, by means of the assumptions (H_1), we have

$$\begin{aligned}& u_0''(t) + M(t)u_0(t) + N(t)u_0'(t) + L(t)(Tu_0)(t) \\ & \quad \leqslant f\big(t, u_0(t), u_0'(t), (Tu_0)(t), (Su_0)(t)\big) + M(t)u_0(t) + N(t)u_0'(t) + L(t)(Tu_0)(t) \\ & \quad \leqslant f\big(t, h(t), h'(t), (Th)(t), (Sh)(t)\big) + M(t)h(t) + N(t)h'(t) + L(t)(Th)(t) \\ & \quad \leqslant f\big(t, v_0(t), v_0'(t), (Tv_0)(t), (Sv_0)(t)\big) + M(t)v_0(t) + N(t)v_0'(t) + L(t)(Tv_0)(t) \\ & \quad \leqslant v_0''(t) + M(t)v_0(t) + N(t)v_0'(t) + L(t)(Tv_0)(t).\end{aligned}$$

Then the normality of P_C implies $\{f(t, h, h', Th, Sh) + M(t)h + N(t)h' + L(t)Th \mid h \in G\}$ is a bounded set in $C[J, E]$. From these, (3.5) and the normality of P_C, we can see that $K \subset G$ is uniformly bounded and equicontinuous on J. Let $C \subset K \subset C[J, E]$ be any countable subset satisfying $\overline{C} = \overline{co}(\{u\} \cup B(C))$ for some $u \in K$. From Lemma 2.3, it follows that $\overline{C(t)} = \overline{co}(\{u(t)\} \cup (BC)(t))$, $t \in J$. From Lemma 2.4, we have

$$\alpha\big((TC)(s)\big) \leqslant k_0 \int_0^t \alpha\big(C(s)\big)\,ds, \tag{3.9}$$

$$\alpha\big((SC)(s)\big) \leqslant h_0 \int_0^a \alpha\big(C(s)\big)\,ds. \tag{3.10}$$

Let $m(t) = \alpha_1(C(t))$, then $m(0) = 0$, $m \in C[J, \mathbf{R}^+]$. Hence, with Lemma 2.7, (H_2), (3.9) and (3.10), we have

$$\begin{aligned}
\alpha\big(C(t)\big) &= \alpha\big(\overline{C(t)}\big) = \alpha\big((BC)(t)\big) \\
&\leqslant \alpha\Bigg(\int_0^t (t-s)\big[f\big(t, C(s), C'(s), (TC)(s), (SC)(s)\big) + M(s)C(s) \\
&\qquad + N(s)C'(s) + L(s)(TC)(s)\big]\,ds\Bigg) \\
&\leqslant 2\int_0^t a\Bigg[(r_1+M)\alpha\big(C(s)\big) + (r_2+N)\alpha\big(C'(s)\big) + (r_3+L)k_0 \int_0^s \alpha\big(C(\tau)\big)\,d\tau \\
&\qquad + r_4 h_0 \int_0^a \alpha\big(C(\tau)\big)\,d\tau\Bigg]\,ds \\
&\leqslant 2a\Bigg[(r_1+M)a\alpha\big(C(t)\big) + (r_2+N)a\alpha\big(C'(t)\big) + (r_3+L)k_0 t \int_0^t \alpha\big(C(s)\big)\,ds \\
&\qquad + r_4 h_0 t \int_0^a \alpha\big(C(s)\big)\,ds\Bigg].
\end{aligned} \tag{3.11}$$

By similar method, we can obtain

$$\begin{aligned}
\alpha\big(C'(t)\big) \leqslant 2\Bigg[&(r_1+M)a\alpha\big(C(t)\big) + (r_2+N)a\alpha\big(C'(t)\big) + (r_3+L)k_0 t \int_0^t \alpha\big(C(s)\big)\,ds \\
&+ r_4 h_0 t \int_0^a \alpha\big(C(s)\big)\,ds\Bigg].
\end{aligned} \tag{3.12}$$

By Lemma 2.2 and (3.11), (3.12), we know that

$$m(t) \leqslant M_1 \int_0^t m(s)\,ds + M_2 t \int_0^t m(s)\,ds + M_3 t \int_0^a m(s)\,ds, \quad t \in J, \tag{3.13}$$

where $M_1 = 2(a+1)(r_1+r_2+M+N)$, $M_2 = 2(a+1)(r_3+L)k_0$, $M_3 = 2(a+1)r_4h_0$. Hence, with Lemma 2.5 and one of the two conditions (i) or (ii) of (H_2) and (3.13), $\alpha_1(C(t)) = 0$, $t \in J$.

Thus, by the Ascoli–Arzela theorem, C is relative compact. It follows from Lemma 2.6 that B has a fixed point $u^* \in K \subset G$, which is a solution of the (IVP) (1.1). This completes the proof of Theorem 3.1. $\square$

By further discussion we can obtain the unique solution and its iterative approximation.

Theorem 3.2. *Let E be a real Banach space and P be a normal cone in E. Assume that conditions (H_1) and (H_3) hold, then (IVP) (1.1) has a unique solution $u^* \in G$. Moreover, for any initial $u_0 \in G$, there exists sequences $\{u_n\} \subset G$, defined as*

$$\begin{aligned} u_n(t) = x_0 + tx_1 + \int_0^t (t-s)\big[& f\big(s, u_{n-1}(s), u'_{n-1}(s), (Tu_{n-1})(s), (Su_{n-1})(s)\big) \\ & + M(s)u_{n-1}(s) + N(s)u'_{n-1}(s) + L(s)(Tu_{n-1})(s) \\ & - M(s)u_n(s) - N(s)u'_n(s) - L(s)(Tu_n)(s)\big]\, ds, \end{aligned} \tag{3.14}$$

which converges uniformly to $u^*(t)$ on J.

Proof. From (H_1) and the proof of Theorem 3.1, we can conclude that the operator $B : G \to G$ defined by (3.5) is continuous and (3.14) can be written as

$$u_n = Bu_{n-1}, \quad n = 1, 2, \ldots. \tag{3.15}$$

We also divide the following proof into three steps.

(I) At first, we will show that (H_3) can imply (H_2). In fact, for any set $B \subset G$, it is easy to know that B is bounded in $C[0,1]$ because of the normality of P. For any $\varepsilon > 0$, from the definition of non-compactness, we know that there exist three kinds of division methods satisfying

$$\begin{aligned} & B = \bigcup_{i=1}^{m_1} B_i, \qquad TB = \bigcup_{j=1}^{m_2} TB_j, \qquad SB = \bigcup_{k=1}^{m_3} SB_k, \\ & d(B_i) \leqslant \alpha(B) + \varepsilon, \quad i = 1, 2, \ldots, m_1, \\ & d(TB_j) \leqslant \alpha(TB) + \varepsilon, \quad j = 1, 2, \ldots, m_3, \\ & d(SB_k) \leqslant \alpha(SB) + \varepsilon, \quad k = 1, 2, \ldots, m_3. \end{aligned}$$

Note that

$$f(t, B, B', TB, SB) = \bigcup_{i=1}^{m_1} \bigcup_{j=1}^{m_2} \bigcup_{k=1}^{m_3} f(t, B_i, B'_i, TB_j, SB_k).$$

Consequently, it follows from (H_3) that

$$\begin{aligned} d\big(f(t, B_i, B'_i, TB_j, SB_k)\big) & \leqslant r_1(t)d(B_i) + r_2(t)d(B'_i) + r_3(t)d(TB_j) + r_4(t)d(SB_k) \\ & \leqslant r_1(t)\alpha(B) + r_2(t)\alpha(B') + r_3(t)\alpha(TB) + r_4(t)\alpha(SB) \\ & \quad + (r_1 + r_2 + r_3 + r_4)\varepsilon. \end{aligned}$$

Hence, by the randomness of ε, we know that

$$\alpha\big(f(t, B, B', TB, SB)\big) \leqslant r_1(t)\alpha(B) + r_2(t)\alpha(B') + r_3(t)\alpha(TB) + r_4(t)\alpha(SB).$$

So, (H_2) holds.

(II) We will prove $\{u_n\}$ converges uniformly to a solution $u^* \in G$ of (IVP) (1.1). In reality, it is easy to see that the normality of P implies the normality of P_C. Moreover, $\{u_n\} \subset G$, so $\{u_n(t)\}$ is uniformly bounded. Form (H_3), we know that $\{f(t, u_n(t), u_n'(t), (Tu_n)(t), (Su_n)(t))\}$ is uniformly bounded on J. Hence, for any $h \in G$,

$$\bigl\{f\bigl(t, h(t), h'(t), (Th)(t), (Sh)(t)\bigr) + M(t)(h-u_n)(t) \\ + N(t)(h-u_n)'(t) + L(t)T(h-u_n)(t)\bigr\}$$

is uniformly bounded on J. Then from (3.14), we obtain the equicontinuity of sequence $\{u_n\}$. Let $m(t) = \alpha_1(\{u_n(t) \mid n = 1, 2, \dots\})$. Note the above conclusion (I), then by the method which was used in proving (3.11)–(3.13), we have

$$m(t) \leqslant M_1 \int_0^t m(s)\,ds + M_2 t \int_0^t m(s)\,ds + M_3 t \int_0^a m(s)\,ds, \quad t \in J, \tag{3.16}$$

where M_i $(i = 1, 2, 3)$ is the same as M_i $(i = 1, 2, 3)$ of (3.13). Hence, by Lemma 2.5, $m(t) = 0$, $t \in J$. Thus by the Ascoli–Arzela theorem $\{x_n\}$ is relatively compact in $C[J, E]$, so there exists subsequence of $\{u_n\}$ which converges uniformly to $u^* \in G$. If we can prove

$$\bigl\|u_n(t) - u_{n-1}(t)\bigr\| \to 0 \quad (n \to \infty),\ t \in J, \tag{3.17}$$

then from the definition of $\{u_n\}$, it is clear that the full sequence $\{u_n(t)\}$ converges uniformly to u^* on J (see [3]). Moreover note (3.15) as well as that the operator B is continuous, it is clear that $u^* \in G$ is a solution of (IVP) (1.1). By (H_3) and (3.14), we know

$$\begin{aligned}
&\bigl\|u_{n+1}(t) - u_n(t)\bigr\| \\
&\quad \leqslant \int_0^t (t-s)\bigl[(r_1+M)\bigl\|u_n(s) - u_{n-1}(s)\bigr\| + (r_2+N)\bigl\|u_n'(s) - u_{n-1}'(s)\bigr\| \\
&\qquad + (r_3+L)\bigl\|T\bigl(u_n(s) - u_{n-1}(s)\bigr)\bigr\| + r_4\bigl\|S\bigl(u_n(s) - u_{n-1}(s)\bigr)\bigr\|\bigr]\,ds \\
&\quad \leqslant a[r_1+M]\int_0^t \bigl\|u_n(s) - u_{n-1}(s)\bigr\|\,ds + a(r_2+N)\int_0^t \bigl\|u_n'(s) - u_{n-1}'(s)\bigr\|\,ds \\
&\qquad + a(r_3+L)k_0 t\int_0^t \bigl\|u_n(s) - u_{n-1}(s)\bigr\|\,ds \\
&\qquad + a r_4 h_0 t\int_0^a \bigl\|u_n(s) - u_{n-1}(s)\bigr\|\,ds,
\end{aligned} \tag{3.18}$$

$$\begin{aligned}
&\bigl\|u_{n+1}'(t) - u_n'(t)\bigr\| \\
&\quad \leqslant [r_1+M]\int_0^t \bigl\|u_n(s) - u_{n-1}(s)\bigr\|\,ds + (r_2+N)\int_0^t \bigl\|u_n'(s) - u_{n-1}'(s)\bigr\|\,ds \\
&\qquad + (r_3+L)k_0 t\int_0^t \bigl\|u_n(s) - u_{n-1}(s)\bigr\|\,ds + r_4 h_0 t\int_0^a \bigl\|u_n(s) - u_{n-1}(s)\bigr\|.
\end{aligned} \tag{3.19}$$

So let

$$m(t) = \max\Big\{\lim_{n\to\infty}\sup\big\|u_{n+1}(t) - u_n(t)\big\|,\ \lim_{n\to\infty}\sup\big\|u'_{n+1}(t) - u'_n(t)\big\|\Big\}$$

and by (3.18), (3.19) and Fatou lemma, we know

$$m(t) \leqslant (a+1)(r_1 + M + r_2 + N)\int_0^t m(s)\,ds + (a+1)(r_3 + L)t\int_0^t m(s)\,ds + (a+1)r_4h_0t\int_0^a m(s)\,ds. \tag{3.20}$$

From Lemma 2.5, we have $m(t) \equiv 0$ for $t \in J$. Then (3.17) holds. All the above (I), (II) imply that $\{u_n\}$ converges to a solution $u^* \in G$ of (IVP) (1.1).

(III) In the end we shall show the uniqueness of solution. Assume that (IVP) (1.1) has two solutions $u, v \in G$. Then u and v both satisfy (3.14). Let $m(t) = \|u(t) - v(t)\|$, then by applying the method used in proving (3.16)–(3.20), $m(t) \equiv 0$, $t \in J$, i.e., $u = v$. This completes the uniqueness proof. □

Remark 1. In this paper, we discussed the initial value problems for second-order nonlinear integro-differential equations where f contains u', Tu, Su and obtained its global solutions and unique solutions as well as its iterative approximation under the conditions.

Remark 2. We can see that Theorem 3.1 is suitable to any measure of non-compactness which equal to the Kuratowski measure of non-compactness from the proof of Theorem 3.1.

Remark 3. The significance of Theorem 3.2 lies in deleting compactness conditions, so it is easier to use and examine the conditions.

4. Application

Consider the IVP of the following nonlinear second-order integro-differential equation

$$\begin{cases} u''(t) = \dfrac{1}{(t+2)^2}u(t) + \dfrac{1}{2(t+2)}u'(t), & 0 \leqslant t \leqslant \dfrac{1}{2}, \\ u(0) = 0, \qquad u'(0) = 0. \end{cases} \tag{4.1}$$

Conclusion. System (4.1) has a unique solution $u^*(t)$ satisfying $2 \leqslant u^*(t) \leqslant (t+2)^2$, $0 \leqslant (u^*)'(t) \leqslant 2(t+2)$, $0 \leqslant t \leqslant \frac{1}{2}$.

Proof. Let $E = C[0, \frac{1}{2}]$ with the norm $\|u\| = \max|u(t)|$ and $P = \{u \in E\colon\ u \geqslant 0\}$. Then P is a normal cone in E. System (4.1) can be regarded as an IVP of the form (1.1) in E. In this situation, $J = [0, \frac{1}{2}]$, $K(t,s) = H(t,s) \equiv 0$, and

$$f(t, u, u') = \frac{1}{(t+2)^2}u(t) + \frac{1}{2(t+2)}u'(t).$$

Obviously, $f(t,u,v)$ is nondecreasing in u and in v. Firstly, let $u_0(t)=2$, $v_0(t)=(t+2)^2$, $t\in J$, then, we have $u_0(t),\ v_0(t)\in C^2[J,E]$ and $u_0(t)\leqslant v_0(t),\ u_0'(t)\leqslant v_0'(t)$. For any $\sigma\in G=\{u\in C^1[J,E]\colon\ u_0\leqslant u\leqslant v_0,\ u_0'\leqslant u'\leqslant v_0'\}$, $t\in J$, we have

$$\begin{aligned}
f(t,\sigma,\sigma') &= \frac{1}{(t+2)^2}\sigma(t)+\frac{1}{2(t+2)}\sigma'(t)\\
&\leqslant \frac{1}{(t+2)^2}(t+2)^2+\frac{1}{2(t+2)}2(t+2)\\
&= 2 = v_0''(t),\\
f(t,\sigma,\sigma') &= \frac{1}{(t+2)^2}\sigma(t)+\frac{1}{2(t+2)}\sigma'(t)\geqslant \frac{1}{(t+2)^2}2\geqslant 0=u_0''(t).
\end{aligned}$$

So, let $M(t)=N(t)=L(t)\equiv 0$, then, condition (H_1) holds. Secondly, for any $s\in J$, $u_i,v_i\in C^2[J,E]$ $(i=1,2)$, we have

$$\bigl\|f(t,u_1,v_1)-f(t,u_2,v_2)\bigr\|\leqslant \frac{1}{(t+2)^2}\|u_1-u_2\|+\frac{1}{2(t+2)}\|v_1-v_2\|.$$

Let $r_1(t)=\frac{1}{(t+2)^2}$, $r_2(t)=\frac{1}{2(t+2)}$, $r_3(t)=r_4(t)\equiv 0$, obviously, $r_1=\frac{1}{4}$, $r_2=\frac{1}{4}$, $r_3=r_4=0$. For $a=\frac{1}{2}$, $M=N=L=0$, $k_0=h_0=0$, we have

$$\begin{aligned}
&\bigl\|f(t,u_1,v_1)-f(t,u_2,v_2)\bigr\|\leqslant r_1(t)\|u_1-u_2\|+r_2(t)\|v_1-v_2\|,\\
&r_1+r_2=\frac{1}{2}>0,\quad a(a+1)\bigl[2(r_1+r_2)\bigr]=\frac{1}{2}\cdot\frac{3}{2}\cdot 2\cdot\frac{1}{2}=\frac{3}{4}<1.
\end{aligned}$$

So, condition (H_3) holds. Therefore, our conclusion follows form Theorem 3.2. The proof is completed. □

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